

# A note on higher-derivative actions for free higher-spin fields

---

Euihun JOUNG      Karapet MKRTCHYAN<sup>1</sup>

*Scuola Normale Superiore and INFN  
Piazza dei Cavalieri 7, 56126 Pisa, Italy*

*E-mail:* [euihun.joung@sns.it](mailto:euihun.joung@sns.it), [karapet.mkrtchyan@sns.it](mailto:karapet.mkrtchyan@sns.it)

**ABSTRACT:** Higher-derivative theories of free higher-spin fields are investigated focusing on their symmetries. Generalizing familiar two-derivative constrained formulations, we first construct *less-constrained* Einstein-like and Maxwell-like higher-derivative actions. Then, we construct Weyl-like actions—the actions admitting *constrained* Weyl symmetries—with different numbers of derivatives. They are presented in a factorized form making use of Einstein-like and Maxwell-like tensors. The last (highest-derivative) member of the hierarchy of the Weyl-like actions coincides with the Fradkin-Tseytlin conformal higher-spin action in four dimensions.

---

<sup>1</sup>On leave from Yerevan Physics Institute.

# 1 Introduction and Summary

Massless higher-spin representations of the Poincaré group require higher-rank tensor gauge fields for their description. However, from a geometrical viewpoint, these fields have an unusual field-theoretical implication since they favor higher derivatives, and for instance the generalized curvature for a rank- $s$  gauge field involves  $s$  derivatives [1]. As a result, the Lagrangian equation of motion constructed from the curvature would propagate not only spin- $s$  modes but also additional ones including ghosts. For spin- $s$  propagation, one has to reduce the number of derivatives to two: this can be done either by imposing constraints on gauge fields and parameters [2], or by introducing auxiliary fields [3, 4] or non-locality [5]. See e.g. [6–8] for recent reviews on higher-spin gauge theories.

Apart from the higher-spin context, higher-derivative theories have been considered mainly for the purpose of renormalizable gravity [9]. Again, the downside is the non-unitary dynamics with propagating ghost modes. Recently higher-derivative theories started to attract renewed interest, thanks to several observations on how to deal with the ghost problem. Among them, let us mention the following key points:

- In low dimensions, such ghosts can become pure gauge restoring unitarity. An important example is the New Massive Gravity [10].<sup>1</sup>
- The solutions of Einstein gravity with negative cosmological constant can be recovered from certain four-derivative gravity theories choosing appropriate boundary conditions (see e.g. [14–17]).

It is worth noticing that the Weyl symmetry underlies the consistency of these four-derivative theories, although the Lagrangian is not Weyl invariant in general. More precisely, in all these theories, the four-derivative part of the action enjoys a Weyl symmetry at least at the linearized level. In a sense, this symmetry removes the massive scalar from the spectrum, leaving only massless and massive spin two modes – which are relatively ghost.<sup>2</sup> It is instructive to see this point in detail. Around a constant-curvature background  $\bar{g}_{\mu\nu}$ , linearized Weyl gravity admits the factorized expression [21]

$$S_1 = \int d^d x \sqrt{-\bar{g}} \left[ G_{\text{lin}}^{\mu\nu} (I_{\text{FP}}^{-1})_{\mu\nu,\rho\sigma} G_{\text{lin}}^{\rho\sigma} + \frac{d-2}{2(d-1)} \Lambda h^{\mu\nu} G_{\mu\nu}^{\text{lin}} \right], \quad (1.1)$$

in terms of the Fierz-Pauli (FP) mass operator

$$(I_{\text{FP}})_{\mu\nu,\rho\sigma} = \bar{g}_{\mu\rho} \bar{g}_{\nu\sigma} - \bar{g}_{\mu\nu} \bar{g}_{\rho\sigma} \quad (1.2)$$

Where  $G_{\mu\nu}^{\text{lin}} = R_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} R + \Lambda \bar{g}_{\mu\nu}$  is the linearized cosmological Einstein tensor. Thanks to the FP operator in  $S_1$ , with the addition of a massless spin two action  $S_2$ :

$$S_2 = m^2 \int d^d x \sqrt{-\bar{g}} h^{\mu\nu} G_{\mu\nu}^{\text{lin}}, \quad (1.3)$$

---

<sup>1</sup> See also [11–13] and references therein for generalizations.

<sup>2</sup> General four-derivative gravity propagates a massless spin two, a massive scalar and a massive spin two ghost [9, 18–20].

the full action can be easily recast into the *difference* between massless and massive spin two actions:

$$S_1 + S_2 = M^2 \int d^d x \sqrt{-g} \left[ \tilde{h}^{\mu\nu} G_{\mu\nu}^{\text{lin}}(\tilde{h}) - \varphi^{\mu\nu} \left\{ G_{\mu\nu}^{\text{lin}}(\varphi) + M^2 (I_{\text{FP}})^{\rho\sigma}_{\mu\nu} \varphi_{\rho\sigma} \right\} \right], \quad (1.4)$$

with  $M^2 = m^2 + \frac{d-2}{2(d-1)} \Lambda$ . The equality in (1.4) can be proved by first performing field redefinition  $\tilde{h}_{\mu\nu} = h_{\mu\nu} - \varphi_{\mu\nu}$  and then solving  $\varphi_{\mu\nu}$  in terms of  $h_{\mu\nu}$ . In three dimensions, the massless spin two does not propagate, and therefore with an appropriate choice of the sign for the kinetic term one can retain only ghost-free massive spin two. On the other hand, in generic dimensional AdS backgrounds, the massless spin two can be selected as the only propagating mode imposing appropriate boundary conditions.

Turning to the case of higher-spin fields, one may wonder whether Weyl(-like) symmetries can still play a similar role in controlling the spectrum of four(or even higher)-derivative theories. Focusing on the free case, the natural generalization of (linearized) diffeomorphisms and Weyl transformations to higher spins would be

$$\delta \varphi_{\mu_1 \dots \mu_s} = \partial_{(\mu_1} \varepsilon_{\mu_2 \dots \mu_s)} + \eta_{(\mu_1 \mu_2} \alpha_{\mu_3 \dots \mu_s)}, \quad (1.5)$$

where  $(\mu_1 \dots \mu_s)$  denotes full symmetrization. In  $d \geq 4$ , a free action possessing the above symmetry involves the square of the higher-spin Weyl tensor (the traceless part of the higher-spin curvature), and thus contains  $2s$  derivatives for a given spin  $s$ . For  $d = 4$ , it coincides with the conformal higher-spin theory<sup>3</sup> of [22–24] (see also [25–29]). Although one can directly consider  $2s$ -derivative theories and study their ghost exorcising mechanisms, one can still wonder whether interesting theories exist whose number of derivatives lies between  $2$  and  $2s$ .

| $\partial^2$ | $\partial^4$ | $\dots$ | $\partial^{2s-2}$ | $\partial^{2s}$ |
|--------------|--------------|---------|-------------------|-----------------|
| Fronsdal     | ?            | $\dots$ | ?                 | Weyl            |

**Table 1.** Hierarchy of higher-derivative actions for higher spins

If these theories exist, we expect that the more derivatives they contain, the more their symmetries are enhanced. The two ends of this hierarchy of theories correspond to the Fronsdal and higher-spin Weyl theories that admit, respectively, constrained gauge symmetries and unconstrained gauge plus Weyl symmetries, while other members are expected to possess symmetries that are intermediate between these.

In the present paper we investigate higher-derivative theories of free higher-spin fields focusing on mainly their symmetries. We begin with familiar two-derivative constrained formulations of higher-spin fields: Fronsdal’s theory [2] and transverse invariant setting of Skvortsov and Vasiliev (SV) [30], recently investigated in a wider context in [31]. Then, increasing the number of derivatives, we identify the Lagrangians compatible with relaxed constraints, and in particular those acquiring Weyl(-like) symmetries.

---

<sup>3</sup> Conformal spin- $s$  actions in  $d$  dimensions contain  $(d + 2s - 4)$ -derivatives. In order to avoid confusion between the conformal actions and the  $2s$ -derivative Weyl squared actions, we call the latter Weyl action, as opposed to conformal action, throughout the present paper.

**Einstein-like actions** We find that increasing the number of derivatives from Fronsdal's setting (doubly traceless field:  $\varphi'' = 0$ , and traceless parameter:  $\varepsilon' = 0$ ), at each step there is unique Lagrangian with higher trace constraints. More precisely, the  $2n$ -derivative Einstein-like actions are determined as

$$\mathcal{G}_{2n} = \int d^d x \, \varphi_{\mu_1 \dots \mu_s} G_{2n}^{\mu_1 \dots \mu_s}, \quad (1.6)$$

where the gauge field is subject to the  $(n+1)$ -th traceless constraint:  $\varphi^{[n+1]} = 0$ , while the gauge parameter to the  $n$ -th traceless constraint:  $\varepsilon^{[n]} = 0$ .

**Maxwell-like actions** On the other hand, departing from the setting of SV (traceless field:  $\varphi' = 0$  and traceless and transverse parameter:  $\varepsilon' = 0 = \partial \cdot \varepsilon$ ), we obtain the  $2n$ -derivative Maxwell-like actions

$$\mathcal{M}_{2n} = \int d^d x \, \varphi_{\mu_1 \dots \mu_s} M_{2n}^{\mu_1 \dots \mu_s}, \quad (1.7)$$

whose fields and parameters are subject to the constraints:  $\varphi^{[n]} = 0$  and  $\varepsilon^{[n]} = 0 = \partial \cdot \varepsilon^{[n-1]}$ . Let us notice that the Maxwell-like actions  $\mathcal{M}_{2n}$  can be regarded as partially gauge fixed versions of the Einstein-like actions  $\mathcal{G}_{2n}$ , as its two-derivative case ( $n = 1$ ).

It is worth noticing that the Einstein-like tensors  $G_{2n}$  and the Maxwell-like tensors  $M_{2n}$  admit a full factorization in terms of  $n$  two-derivative operators, whose first factors are the Fronsdal and Maxwell ones. It was shown in [32] that the equations  $G_{2n} \approx 0$  and  $M_{2n} \approx 0$  are equivalent respectively to the equations  $R^{[n]} \approx 0$  and  $\partial \cdot R^{[n-1]} \approx 0$  where  $R$  is the linearized higher-spin curvature. These results accord with the conjecture based on the cohomological analysis of the equations of motion (see the footnote 25 in [6]).

The Einstein-like actions exist up to  $2\lfloor s/2 \rfloor$  derivatives while the Maxwell-like ones to  $2\lfloor (s+1)/2 \rfloor$  derivatives. Hence, they can cover only half of Table 1. The last member of each set (Einstein-like for even spin and Maxwell-like for odd spin) admits an unconstrained gauge symmetry and is essentially the local counterpart of the unconstrained non-local Lagrangian of [5]. Although there is no generic symmetry enhancement beyond  $s$  derivatives, we show that for  $2s + d - 4$  derivatives Einstein-(Maxwell-)like actions acquire Weyl symmetry. They coincide with the linearized conformal higher-spin action since the latter has the same number of derivatives and the same symmetries.

**Weyl-like actions** The higher-spin Weyl action involves  $2s$  derivatives and does not require any constraint on its gauge field and parameters. It turns out that Weyl symmetry can be realized also in lower derivative cases (from four to  $2s - 2$  derivatives) with constrained parameters. We identify Lagrangians possessing Weyl symmetries with parameters, subject to differential constraints. The Weyl-like actions are split into two classes: the first involves  $4n$  derivatives while the other involves  $4n + 2$  derivatives.

- The  $4n$ -derivative Weyl-like actions are obtained through Einstein-like tensors as

$$\mathcal{W}_{4n} = \int d^d x \, G_{2n}^{\mu_1 \dots \mu_s} (I_F^{-1})_{\mu_1 \dots \mu_s, \nu_1 \dots \nu_s} G_{2n}^{\nu_1 \dots \nu_s}, \quad (1.8)$$

where  $I_F$  is the mass operator for higher-spin fields considered by Francia [33]. Let us notice the similarity between these Lagrangians (1.8) and the gravity case (1.1). Both cases admit a factorization in terms of two Einstein(-like) tensors with the inverse of the Fierz-Pauli (Francia) mass operator. The constraints on gauge fields and parameters follow those of the  $2n$ -derivative Einstein-like action  $\mathcal{G}_{2n}$ , while the Weyl parameters are subject to new types of constraints – traceless and  $(2n - 1)$ -th divergence-less conditions:  $\alpha' = 0 = (\partial \cdot)^{2n-1} \alpha$ .

- The  $(4n + 2)$ -derivative Weyl-like actions are determined using both Einstein-like and Maxwell-like tensors, as

$$\mathcal{W}_{4n+2} = \int d^d x \, G_{2n}^{\mu_1 \dots \mu_s} (I_F^{-1})_{\mu_1 \dots \mu_s, \nu_1 \dots \nu_s} M_{2n+2}^{\nu_1 \dots \nu_s}. \quad (1.9)$$

In this case, the constraints on the gauge fields and parameters follow those of the  $(2n + 2)$ -derivative Maxwell-like action  $\mathcal{M}_{2n+2}$ , while the Weyl parameters satisfy  $\alpha' = 0 = (\partial \cdot)^{2n} \alpha$ .

Note that these Weyl-like actions coincide with the higher-spin Weyl action when  $n = s/2$  or  $(s - 1)/2$ , providing a new expression for the Weyl squared action.

The organization of this paper is as follows. In Section 2, we review the generalized Christoffel symbols à la de Wit and Freedman and construct higher-derivative Einstein-like and Maxwell-like actions for higher-spin fields. In Section 3, we turn to Weyl-like actions and show that they are given in terms of Einstein-like and Maxwell-like tensors with a factor that is an inverse mass operator. Section 4 contains the discussions on the (A)dS deformations of the Weyl(-like) actions and their expected properties, as well as some issues on the interacting cases. Finally, Appendices A and B provide the explicit form of the spin three Weyl action and the spectrum analysis of the four-derivative Einstein (or Maxwell)-like action.

## 2 Higher-derivative gauge invariant actions

In this section we discuss higher-derivative actions which admit only gauge symmetries (gauge fields transforming with the gradient of the parameter). They are constructed using the deWit-Freedman generalized Christoffel symbols (GCS) [1], which we review briefly in the following subsection. Before starting our discussion, let us introduce the generating function (or auxiliary variable) notation for the higher-spin fields

$$\varphi^{(s)}(x, u) = \frac{1}{s!} \varphi_{\mu_1 \dots \mu_s}(x) u^{\mu_1} \dots u^{\mu_s}, \quad (2.1)$$

which we shall use throughout the present paper. In this notation, the gauge and Weyl transformations (1.5) are expressed as

$$\delta \varphi^{(s)}(x, u) = u \cdot \partial_x \varepsilon^{(s-1)}(x, u) + u^2 \alpha^{(s-2)}(x, u), \quad (2.2)$$

where  $\varepsilon^{(s-1)}(x, u)$  and  $\alpha^{(s-2)}(x, u)$  are the generating functions of the gauge parameters  $\varepsilon_{\mu_1 \dots \mu_{s-1}}$  and the Weyl parameters  $\alpha_{\mu_1 \dots \mu_{s-2}}$  defined analogously to (2.1). Moreover, the actions of free higher-spin fields will be represented using the scalar product

$$\begin{aligned} \langle\langle \phi^{(s)} | \psi^{(s)} \rangle\rangle &:= \int d^d x \, e^{\partial_{u_1} \cdot \partial_{u_2}} \phi^{(s)}(x, u_1) \psi^{(s)}(x, u_2) \Big|_{u_1=u_2=0} \\ &= \int d^d x \, \frac{1}{s!} \phi_{\mu_1 \dots \mu_s}(x) \psi^{\mu_1 \dots \mu_s}(x). \end{aligned} \quad (2.3)$$

## 2.1 deWit-Freedman generalized Christoffel symbols

The Christoffel symbol of gravity, linearized around the flat background so that  $g_{\mu\nu} = \eta_{\mu\nu} - h_{\mu\nu}$ , is given by

$$\Gamma_{\mu\nu, \rho} = \frac{1}{2} (\partial_\rho h_{\mu\nu} - \partial_\mu h_{\nu\rho} - \partial_\nu h_{\mu\rho}). \quad (2.4)$$

Rewriting this in the auxiliary-variable notation gives

$$\Gamma^{(s,1)}(x, u, v) = \Gamma_{\mu\nu, \rho}(x) u^\mu u^\nu v^\rho = (v \cdot \partial_x - u \cdot \partial_x v \cdot \partial_u) \varphi^{(s)}(x, u), \quad (2.5)$$

with  $s = 2$ . The general spin case of (2.5) corresponds to the first member of the GCS hierarchy, and its variation under the gauge transformation results in the double gradient as

$$\delta \Gamma^{(s,1)}(x, u, v) = -(u \cdot \partial_x)^2 (v \cdot \partial_u) \varepsilon^{(s-1)}(x, u), \quad (2.6)$$

while the other members of the hierarchy can be determined recursively by

$$\Gamma^{(s,r)}(x, u, v) = (v \cdot \partial_x - \frac{1}{r} u \cdot \partial_x v \cdot \partial_u) \Gamma^{(s,r-1)}(x, u, v), \quad (2.7)$$

in such a way that their gauge variations give rise to the multiple gradients

$$\delta \Gamma^{(s,r)}(x, u, v) = \frac{(-1)^r}{r!} (u \cdot \partial_x)^{r+1} (v \cdot \partial_u)^r \varepsilon^{(s-1)}(x, u). \quad (2.8)$$

In this way, the last member of the hierarchy, called deWit-Freedman curvature,

$$\Gamma^{(s,s)}(x, u, v) = \frac{1}{s!} (u \cdot \partial_x v \cdot \partial_w - v \cdot \partial_x u \cdot \partial_w)^s \varphi^{(s)}(x, w), \quad (2.9)$$

becomes gauge invariant without any constraint on gauge field or parameter.

In the following subsections, we construct higher-derivative gauge invariant actions making use of these GCS. Our construction essentially follows that of [5], although the context here is different.

## 2.2 Einstein-like actions

From the gauge transformations (2.8), one can see that multiple  $v$ -traces of the symbols

$$F_{2n}^{(s)}(x, u) := (\partial_v^2)^n \Gamma^{(s,2n)}(x; u, v), \quad (2.10)$$

transform into the multiple traces of the gauge parameters

$$\delta F_{2n}^{(s)}(x, u) = (u \cdot \partial_x)^{2n+1} (\partial_u^2)^n \varepsilon^{(s-1)}(x, u). \quad (2.11)$$

For  $n = 1$ , the object  $F_2^{(s)}$  coincides with the spin- $s$  analogue of the Ricci tensor, given by the Fronsdal's operator  $\mathcal{F}_2$

$$F_2^{(s)}(x, u) = \mathcal{F}_2 \varphi^{(s)}(x, u), \quad \mathcal{F}_2 := \partial_x^2 - u \cdot \partial_x \partial_u \cdot \partial_x + \frac{1}{2} (u \cdot \partial_x)^2 \partial_u^2, \quad (2.12)$$

and is invariant under the gauge transformations generated by traceless parameters. For general  $n$ , the  $F_{2n}^{(s)}$ 's are invariant under gauge transformations with parameters subject to higher-trace constraints

$$(\partial_u^2)^n \varepsilon^{(s-1)}(x, u) = 0. \quad (2.13)$$

For the subsequent analysis, we provide another useful expression for  $F_{2n}^{(s)}$ , which can be obtained using the identity relating  $\Gamma^{(s,r)}$  and  $\Gamma^{(s,r-2)}$ :

$$\partial_v^2 \Gamma^{(s,r)}(x, u, v) = \mathcal{F}_r \Gamma^{(s,r-2)}(x, u, v). \quad (2.14)$$

Here we have introduced the generalized Fronsdal operators  $\mathcal{F}_r$  as

$$\begin{aligned} \mathcal{F}_r &:= (\partial_x - \frac{1}{r} u \cdot \partial_x \partial_u) \cdot (\partial_x - \frac{1}{r-1} u \cdot \partial_x \partial_u) \\ &= \partial_x^2 - \frac{2}{r} u \cdot \partial_x \partial_u \cdot \partial_x + \frac{1}{r(r-1)} (u \cdot \partial_x)^2 \partial_u^2. \end{aligned} \quad (2.15)$$

After iterations, the Ricci-like tensors  $F_{2n}^{(s)}$  can be factorized as

$$F_{2n}^{(s)}(x, u) = \mathcal{F}_{2n} \mathcal{F}_{2n-2} \cdots \mathcal{F}_4 \mathcal{F}_2 \varphi^{(s)}(x, u), \quad (2.16)$$

where the order of the  $\mathcal{F}_{2r}$ 's is important since they do not commute with each other. The  $F_{2n}^{(s)}$ 's satisfy higher-derivative analogues of the Bianchi-like identities

$$\left( \partial_u \cdot \partial_x - \frac{1}{2(n+r)} u \cdot \partial_x \partial_u^2 \right) (\partial_u^2)^r F_{2n}^{(s)} = 0, \quad [r = 0, 1, \dots, n-1], \quad (2.17)$$

on the space of the  $(n+1)$ -th traceless gauge field:

$$(\partial_u^2)^{n+1} \varphi^{(s)}(x, u) = 0. \quad (2.18)$$

Let us now construct the actions giving rise to the equations  $F_{2n}^{(s)} = 0$ . In the two-derivative ( $n = 1$ ) case, the spin- $s$  Einstein tensor  $G_2^{(s)}$  can be obtained from the Ricci tensor by a trace modification:

$$G_2^{(s)} = \mathcal{I}_2 F_2^{(s)}, \quad \mathcal{I}_2 = 1 - \frac{1}{4} u^2 \partial_u^2. \quad (2.19)$$

Imposing the doubly-traceless constraint on fields, the Einstein tensor becomes self-adjoint and gives the Fronsdal Lagrangian:  $\mathcal{G}_2 = \langle\langle \varphi^{(s)} | G_2^{(s)} \rangle\rangle$ . For the higher-derivative ( $n \geq 2$ ) cases, one can also consider the Einstein-like tensors  $G_{2n}^{(s)}$  by appropriately modifying the traces of the Ricci-like tensors  $F_{2n}^{(s)}$  as

$$G_{2n}^{(s)} = \mathcal{I}_{2n} F_{2n}^{(s)}, \quad (2.20)$$

where  $\mathcal{I}_{2n}$  is an operator of form  $\sum_k a_k (u^2)^k (\partial_u^2)^k$ . One can determine  $\mathcal{I}_{2n}$  either by requiring the gauge invariance of the action or the self-adjointness of  $G_{2n}^{(s)}$ . The gauge

invariance requires that the divergence of  $G_{2n}^{(s)}$  gives the Bianchi-like identities (2.17), and this gives the condition

$$\partial_u \cdot \partial_x \mathcal{I}_{2n} = \tilde{\mathcal{I}}_{2n} \left( \partial_u \cdot \partial_x - \frac{1}{2n} u \cdot \partial_x \partial_u^2 \right) \quad (2.21)$$

on the operator  $\mathcal{I}_{2n}$ , where  $\tilde{\mathcal{I}}_{2n}$  is an operator of the same type as  $\mathcal{I}_{2n}$ . The above condition fixes uniquely the operator as

$$\mathcal{I}_{2n} = \sum_{r=0}^{\lfloor s/2 \rfloor} \frac{1}{r! [n]_r} \left(-\frac{1}{4}\right)^r (u^2)^r (\partial_u^2)^r, \quad (2.22)$$

where

$$[a]_r := a(a-1) \cdots (a-r+1) \quad (2.23)$$

are descending Pochhammer symbols. As one can see from the pole arising when  $r = n+1$ , the  $(n+1)$ -th traceless condition (2.18) is indispensable for the Einstein-like actions

$$\mathcal{G}_{2n}[\varphi^{(s)}] = \langle\langle \varphi^{(s)} | G_{2n}^{(s)} \rangle\rangle. \quad (2.24)$$

To recapitulate, these actions are compatible with the gauge fields subject to the  $(n+1)$ -th traceless constraint (2.18), and they are invariant under the gauge transformations generated by the  $n$ -th traceless gauge parameters (2.13).

### 2.3 Maxwell-like actions

Coming back to the construction (2.10), instead of taking only traces one can also take a divergence, obtaining<sup>4</sup>

$$L_{2n}^{(s)}(x, u) := \partial_v \cdot \partial_x (\partial_v^2)^{n-1} \Gamma^{(s, 2n-1)}(x; u, v), \quad (2.27)$$

then it transforms under the gauge variation as

$$\delta L_{2n}^{(s)}(x, u) = (u \cdot \partial_x)^{2n} \partial_u \cdot \partial_x (\partial_u^2)^{n-1} \varepsilon^{(s-1)}(x, u). \quad (2.28)$$

For  $n = 1$ , the object  $L_2^{(s)}$  coincides with the spin- $s$  tensor of transverse-invariant theories [30, 31], defined in terms of Maxwell operator  $\mathcal{L}$ ,

$$L_2^{(s)}(x, u) = \mathcal{L} \varphi^{(s)}(x, u), \quad \mathcal{L} := \partial_x^2 - u \cdot \partial_x \partial_u \cdot \partial_x, \quad (2.29)$$

---

<sup>4</sup> More generally, one may consider

$$(\partial_x \cdot \partial_v)^m (\partial_v^2)^n \Gamma^{(s, 2n+m)}(x; u, v), \quad (2.25)$$

whose gauge symmetry requires the constraints:

$$(\partial_x \cdot \partial_u)^m (\partial_u^2)^n \varepsilon^{(s-1)}(x, u) = 0. \quad (2.26)$$

Let us note that when  $n = 0$ , the tensor (2.25) becomes self-adjoint without any constraint on the field so directly provides the Lagrangian of the theory, analogously to the *reducible* transverse-invariant theory of [31] with  $m = 1$  and the curvature squared theory with  $m = s$  which is the local counterpart of [34].



and it is invariant under the gauge transformations generated by transverse parameter. For general  $n$ , the SV-like tensors  $L_{2n}^{(s)}$ 's are invariant under the gauge transformations with the parameters subject to

$$\partial_u \cdot \partial_x (\partial_u^2)^{n-1} \varepsilon^{(s-1)}(x, u) = 0. \quad (2.30)$$

Moreover, as in the case of  $F_{2n}^{(s)}$ , they can be factorized as

$$\begin{aligned} L_{2n}^{(s)}(x, u) &= \mathcal{F}_{2n-1} \mathcal{F}_{2n-3} \cdots \mathcal{F}_5 \mathcal{F}_3 \mathcal{L} \varphi^{(s)}(x, u) \\ &= \left( \partial_x^2 - \frac{1}{2n-1} u \cdot \partial_x \partial_x \cdot \partial_u \right) F_{2n-2}(x, u). \end{aligned} \quad (2.31)$$

From the second expression, one can see that on the space of  $n$ -th traceless fields, such that

$$(\partial_u^2)^n \varphi^{(s)}(x, u) = 0, \quad (2.32)$$

they satisfy the Bianchi-like identities:

$$\left( \partial_x \cdot \partial_u - \frac{1}{2(n+r)} u \cdot \partial_x \partial_u^2 \right) (\partial_u^2)^r L_{2n}(x, u) = 0 \quad [r = 0, 1, \dots, n-1]. \quad (2.33)$$

To comply with the  $n$ -th traceless constraint on the field, it is necessary to impose the same condition on the gauge parameter:

$$(\partial_u^2)^n \varepsilon^{(s-1)}(x, u) = 0. \quad (2.34)$$

Finally the action leading to the equation  $L_{2n}^{(s)} = 0$  can be determined as

$$\mathcal{M}_{2n}[\varphi^{(s)}] = \langle\langle \varphi^{(s)} | M_{2n}^{(s)} \rangle\rangle, \quad M_{2n}^{(s)} = \mathcal{I}_{2n} L_{2n}^{(s)}, \quad (2.35)$$

thanks to the Bianchi-like identities (2.33). To recapitulate, these actions are compatible with gauge fields subject to the  $n$ -th traceless constraint (2.32) and are invariant under the gauge transformations generated by parameters subject to the constraints (2.30) and (2.34). One may regard these actions  $\mathcal{M}_{2n}$  as partially gauge fixed versions of the Einstein-like actions  $\mathcal{G}_{2n}$ , with gauge fixing (2.18) to (2.32). As mentioned in Introduction, the equations of motion of these theories are shown to be equivalent to ones involving higher-spin curvature [32].

### 3 Weyl-like actions

In the previous section, allowing higher derivatives in the quadratic action we have constructed the  $2n$ -derivative Einstein-like and Maxwell-like actions  $\mathcal{G}_{2n}$  and  $\mathcal{M}_{2n}$ . Starting from  $n = 1$  and increasing the number  $n$ , there is an enhancement of the gauge symmetries due to the weakening of the constraints (2.13) and (2.30, 2.34) imposed on the gauge parameters. However, when  $2n \geq s$ , these constraints are completely removed, so that the actions do not acquire any additional symmetry in general. In other words, the hierarchy of higher-derivative Einstein/Maxwell-like actions with constrained gauge symmetries covers only half of Table 1. It is worth noticing that the  $(d + 2s - 4)$ -derivative Einstein- and

Maxwell-like actions are exceptions from this point of view. In fact, the trace modifier  $\mathcal{I}_{2n}$  in  $\mathcal{G}_{2n}$  and  $\mathcal{M}_{2n}$  become the trace projector when  $2n = d + 2s - 4$ :

$$\mathcal{I}_{d+2s-4} u^2 \alpha^{(s-2)} = 0. \quad (3.1)$$

Therefore, these actions acquire additional Weyl symmetry and coincide with the conformal higher-spin action.

On the other hand, the existence of the  $2s$ -derivative higher-spin Weyl action suggests that there might be another hierarchy of actions with constrained Weyl (and gauge) symmetries. As discussed in the Introduction, the spin-two Weyl action can be written as a square of the Einstein tensor. This gives a crucial hint that Weyl-like actions can be obtained as products of the Einstein-like or Maxwell-like tensors. In the following, we present two classes of Weyl-like actions: the first one involves  $4n$  derivatives while the other  $4n + 2$  derivatives.

### 3.1 $4n$ -derivative Weyl-like actions

#### Four-derivative case

Let us begin with the linearized Weyl gravity action, which admits a factorized expression:

$$\mathcal{W}_4[\varphi^{(2)}] = \langle\langle G_2^{(2)} \mid \left(1 - \frac{1}{2(d-1)} u^2 \partial_u^2\right) G_2^{(2)} \rangle\rangle. \quad (3.2)$$

Generalizing the spin-2 Einstein tensor  $G_2^{(2)}$  to the spin- $s$  one  $G_2^{(s)}$ , we consider the ansatz:

$$\mathcal{W}_4[\varphi^{(s)}] = \langle\langle G_2^{(s)} \mid \mathcal{A}_2 G_2^{(s)} \rangle\rangle, \quad \mathcal{A}_2 = 1 + a u^2 \partial_u^2, \quad (3.3)$$

for the higher-spin action with Weyl symmetry. The form of the ansatz through  $G_2^{(s)}$  already guarantees the constrained gauge symmetry when implemented by the Fronsdal constraints (doubly-traceless/traceless constraint on gauge field/parameter). Turning to the Weyl symmetry, we first notice that its parameter must be traceless,

$$\partial_u^2 \alpha^{(s-2)}(x, u) = 0, \quad (3.4)$$

for compatibility with the doubly-traceless field. An explicit computation then shows that the Weyl symmetry arises for a special value of  $a$  in (3.3),

$$a = -\frac{1}{2(d+2s-5)}, \quad (3.5)$$

which generalizes the spin-2 case – see the coefficient in (3.2). The crucial novelty of the higher-spin case (3.3) with respect to the linearized Weyl gravity (3.2) is that the Weyl symmetry is, in fact, constrained with the *transverse* constraint

$$\partial_x \cdot \partial_u \alpha^{(s-2)}(x, u) = 0. \quad (3.6)$$

Let us remind the reader that an analogous constraint has been considered for the *gauge parameter* in the transverse-invariant theories of [30, 31].

To recapitulate, we have found that a four-derivative Weyl-like action does exist for Fronsdal fields of general spin with Weyl symmetry parameter subject to differential constraint (3.6). One can obtain the same result starting from the Fronsdal constraints but without assuming the form of the action: the action (3.3) with a (3.5) is the unique four-derivative action acquiring Weyl symmetry. In Section 4, we show that the spectrum of this action consists of two massless spin  $s$  (relatively ghost) and a massless spin  $s - 1$  modes, in analogy with the Weyl gravity.

### General case

The form of (3.3) is suggestive, so that we can now generalize it to higher derivative cases. Replacing the two-derivative Einstein tensor  $G_2^{(s)}$  with the  $2n$ -derivative one  $G_{2n}^{(s)}$ , let us consider the ansatz:

$$\mathcal{W}_{4n}[\varphi^{(s)}] = \langle\langle G_{2n}^{(s)} | \mathcal{A}_{2n} G_{2n}^{(s)} \rangle\rangle, \quad (3.7)$$

where  $\mathcal{A}_{2n}$  is the trace modifier to be determined requiring Weyl invariance. As in the four-derivative case, the constraints imposed on gauge field and parameter follow those of the Einstein-like action  $\mathcal{G}_{2n}$ :  $(n + 1)$ -th/ $n$ -th traceless gauge field/parameter. Moreover, considering the Weyl symmetry, its parameter is subject to the  $n$ -th traceless constraint:

$$(\partial_u^2)^n \alpha^{(s-2)}(x, u) = 0, \quad (3.8)$$

for compatibility with the  $(n + 1)$ -th traceless gauge field.

The question we want to turn to is whether the action (3.7), with an appropriate choice of  $\mathcal{A}_{2n}$ , can admit a Weyl symmetry. To answer it, we first compute the Weyl variation of the action, then get an equation for the Weyl parameter  $\alpha^{(s-2)}$  that eliminates the variation. This equation defines a constraint for  $\alpha^{(s-2)}$  and depends on the form of the operator  $\mathcal{A}_{2n}$ . Hence, the point is whether some operator  $\mathcal{A}_{2n}$  can lead to a reasonable constraint on the Weyl parameter  $\alpha^{(s-2)}$ .

Computing the variation of (3.7) under the  $\alpha^{(s-2)}$  transformation, using (2.20), gives

$$\delta_\alpha \mathcal{W}_{4n}[\varphi^{(s)}] = 2 \langle\langle G_{2n}^{(s)} | \mathcal{C}_{2n} \delta_\alpha F_{2n}^{(s)} \rangle\rangle, \quad (3.9)$$

where  $\mathcal{C}_{2n}$  is given by

$$\mathcal{C}_{2n} := \mathcal{A}_{2n} \mathcal{I}_{2n} = \sum_{k=0}^n c_k (u^2)^k (\partial_u^2)^k. \quad (3.10)$$

Since  $\mathcal{I}_{2n}$  is fixed and invertible (for  $2n \neq d + 2s - 4$ ), determining  $\mathcal{C}_{2n}$  is equivalent to determining  $\mathcal{A}_{2n}$ . In the following, we first compute the Weyl variation of the Ricci-like tensor  $F_{2n}^{(s)}$ , and then simplify the expression for  $\mathcal{C}_{2n} \delta_\alpha F_{2n}^{(s)}$ :

- For the computation of  $\delta_\alpha F_{2n}^{(s)}$ , we make use of the identities

$$\mathcal{F}_{2r} u^2 = u^2 \mathcal{F}_{2r} + \frac{d+2}{r(2r-1)} (u \cdot \partial_x)^2, \quad (3.11)$$

$$\mathcal{F}_{2r} (u \cdot \partial_x)^k = \frac{(2r-k)(2r-k-1)}{2r(2r-1)} (u \cdot \partial_x)^k \mathcal{F}_{2r-k}, \quad (3.12)$$

together with the Bianchi-like identities (2.17). Employing these, respectively yields the Weyl variation of the Ricci-like tensors,

$$\begin{aligned}\delta_\alpha F_{2n}^{(s)} &= \mathcal{F}_{2n} \cdots \mathcal{F}_2 u^2 \alpha^{(s-2)} \\ &= \left[ -\frac{1}{2n-1} (u \cdot \partial_x)^2 \left\{ 2(n-\tau) + \frac{1}{2(n-1)} u^2 \partial_u^2 \right\} + u^2 \partial_x^2 \right] F_{2(n-1)}^{(s-2)}(\alpha),\end{aligned}\quad (3.13)$$

where  $\tau = (d + 2s - 4)/2$  and

$$F_{2(n-1)}^{(s-2)}(\alpha) := \mathcal{F}_{2(n-1)} \cdots \mathcal{F}_2 \alpha^{(s-2)}. \quad (3.14)$$

- Next, we express  $\mathcal{C}_{2n} \delta_\alpha F_{2n}^{(s)}$  in the form of a normal ordered operator acting on  $F_{2(n-1)}^{(s-2)}(\alpha)$ , using again the Bianchi-like identities (2.17) together with the identity:

$$\begin{aligned}(\partial_u^2)^n (u^2)^m &= \sum_{k=0}^{\min\{n,m\}} 2^k k! \binom{n}{k} \binom{m}{k} \frac{[d + 2u \cdot \partial_u + 2(n-m+k-1)]!!}{[d + 2u \cdot \partial_u + 2(n-m-1)]!!} \times \\ &\quad \times (u^2)^{m-k} (\partial_u^2)^{n-k},\end{aligned}\quad (3.15)$$

which can be proved by induction. One then finds:

$$\begin{aligned}\mathcal{C}_{2n} \delta_\alpha F_{2n}^{(s)} &= \sum_{k=0}^n \frac{n}{(2n-1)(n-k+1)} \times \\ &\quad \times (u^2)^k [c_k^\sharp \partial_x^2 - c_k^\flat (u \cdot \partial_x)^2 \partial_u^2] (\partial_u^2)^{k-1} F_{2(n-1)}^{(s-2)}(\alpha),\end{aligned}\quad (3.16)$$

where the  $c_k^\sharp$  and  $c_k^\flat$  are coefficients given in terms of the  $c_k$ 's

$$\begin{aligned}c_k^\sharp &= (2n - 2k + 1) c_{k-1} + 4k(n - k + 1)(2\tau - 2k + 1) c_k, \\ c_k^\flat &= \frac{1}{n-k} [c_{k-1} + 4(n - k + 1)(n + k - \tau) c_k].\end{aligned}\quad (3.17)$$

The variation (3.16) contains two types of terms: the first type (with coefficients  $c_k^\sharp$ ) does not involve any overall gradient operators, while the second (with coefficients  $c_k^\flat$ ) does involve an overall double gradient operator. Since this Weyl variation is to be contracted with the Einstein-like tensor  $G_{2n}^{(s)}$ , there is a chance that the terms of the second type vanish due to the divergence-free nature of  $G_{2n}^{(s)}$  (or equivalently due to the Bianchi-like identities). On the other hand, the terms of the first type have no chance of vanishing by themselves, and therefore, we require the  $c_k^\sharp$ 's to vanish, which gives the following recurrence relation on the coefficients  $c_k$ 's:

$$c_k^\sharp = 0 \quad \Rightarrow \quad (2n - 2k + 1) c_{k-1} + 4k(n - k + 1)(2\tau - 2k + 1) c_k = 0. \quad (3.18)$$

This equation (with the choice  $c_0 = 1$ ) uniquely determines the trace modifier  $\mathcal{C}_{2n}$ , and consequently  $\mathcal{A}_{2n}$ .

After fixing  $c_k^\sharp = 0$ , the Weyl variation is given by the  $c_k^\flat$  terms with an overall double gradient. One can integrate by parts one of the gradient operators and get a divergence of the Einstein-like tensor,  $\partial_u \cdot \partial_x G_{2n}^{(s)}$ , which vanishes only when it is contracted with a  $n$ -th

traceless tensor. Hence, Weyl invariance imposes the condition that the  $n$ -th trace of the left-over part (after integrating by part one gradient) be zero:

$$(\partial_u^2)^n u \cdot \partial_x \left[ \sum_{k=0}^n \frac{c_k^b}{n-k+1} (u^2)^k (\partial_u^2)^k \right] F_{2(n-1)}^{(s-2)}(\alpha) = 0. \quad (3.19)$$

Due to the  $n$ -th traceless constraint on  $\alpha^{(s-2)}$ , the above condition reduces to the differential constraint

$$\partial_u \cdot \partial_x (\partial_u^2)^{n-1} \mathcal{F}_{2(n-1)} \cdots \mathcal{F}_2 \alpha^{(s-2)} = 0. \quad (3.20)$$

For the four-derivative ( $n = 1$ ) case, this constraint reduces to the transversality condition (3.6), while for the other values of  $n$  it gives a rather unusual type of constraint containing Fronsdal-like operators. In fact, for the correct analysis of the constraint, one should take into account the gauge-for-gauge symmetry:

$$\delta \varepsilon^{(s-1)}(x, u) = u^2 \beta^{(s-3)}(x, u), \quad \delta \alpha^{(s-2)}(x, u) = -u \cdot \partial_x \beta^{(s-3)}(x, u), \quad (3.21)$$

possessed by the gauge plus Weyl transformations (2.2). Here  $\beta^{(s-3)}$  is the gauge-for-gauge parameter and satisfies the  $(n-1)$ -th traceless constraint:  $(\partial_u^2)^{n-1} \beta^{(s-3)} = 0$ . The importance of the gauge-for-gauge transformation (3.21) is that using it one can always make the Weyl parameter  $\alpha^{(s-2)}$  traceless:

$$\partial_u^2 \alpha^{(s-2)}(x, u) = 0. \quad (3.22)$$

In other words, the trace part of Weyl transformation can always be expressed as a *gauge* transformation. Let us notice also that eq. (3.20) is invariant under the transformation (3.21) for  $\alpha^{(s-2)}$ . Taking into account the condition (3.22), the unusual constraint (3.20) reduces to the requirement of vanishing multiple divergence:

$$(\partial_x \cdot \partial_u)^{2n-1} \alpha^{(s-2)}(x, u) = 0. \quad (3.23)$$

### 3.2 $(4n+2)$ -derivative Weyl-like actions

#### Six-derivative case

Let us begin with the six-derivative spin-3 Weyl action (see Appendix A). Differently from linearized Weyl gravity, it is not given as an Einstein tensor squared but admits another type of factorization:

$$\mathcal{W}_6[\varphi^{(3)}] = \langle\langle F_2^{(3)} \mid \left[ \partial_x^2 - \frac{d-2}{6(d+1)} u \cdot \partial_x \partial_x \cdot \partial_u - \frac{d+4}{12(d+1)} u^2 \partial_x^2 \partial_u^2 \right] F_2^{(3)} \rangle\rangle. \quad (3.24)$$

Generalizing the spin-3 Ricci tensor  $F_2^{(3)}$  to the spin- $s$  one  $F_2^{(s)}$ , we consider for the Weyl invariant action the ansatz

$$\mathcal{W}_6[\varphi^{(s)}] = \langle\langle F_2^{(s)} \mid \left[ \partial_x^2 + a u \cdot \partial_x \partial_x \cdot \partial_u + b u^2 \partial_x^2 \partial_u^2 \right] F_2^{(s)} \rangle\rangle, \quad (3.25)$$

where  $\varphi^{(s)}$  is doubly traceless. Note that this ansatz is the most general one satisfying *i*) manifest self-adjoint-ness and *ii*) factorization in terms of two  $F_2^{(s)}$ 's. More precisely, there

can be other two-derivative operators inside of the square bracket in (3.25), but they can be all replaced with the actual ones by the virtue of Bianchi-like identities.

The traceless gauge symmetry of (3.25) is ensured by the presence of the Fronsda tensor  $F_2^{(s)}$ . On the other hand, as one can check by explicit computations, the Weyl symmetry arises only for

$$a = -\frac{d+2s-8}{6(d+2s-5)}, \quad b = -\frac{d+2s-2}{12(d+2s-5)}, \quad (3.26)$$

with a parameter subject to the traceless and doubly transverse constraints:

$$\partial_u^2 \alpha^{(s-2)}(x, u) = 0, \quad (\partial_x \cdot \partial_u)^2 \alpha^{(s-2)}(x, u) = 0. \quad (3.27)$$

Notice that the above constraints coincide with the formal 6-derivative interpolation of the previously found  $4n$ -derivative constraints (3.22, 3.23), obtained for  $n = \frac{3}{2}$ .

Turning back to the expression (3.25), one may wonder whether it can be recast into a simpler form as in the four-derivative case. Given that it involves six-derivatives, the action cannot be written as a square of Einstein-like or Maxwell-like tensors, but it can, in fact, be expressed as a product between Einstein and four-derivative Maxwell-like tensors as

$$\mathcal{W}_6[\varphi^{(s)}] = \langle\langle G_2^{(s)} \mid \left(1 - \frac{1}{2(d+2s-5)} u^2 \partial_u^2\right) M_4^{(s)} \rangle\rangle. \quad (3.28)$$

Let us notice that the trace modifier lying between the Einstein and Maxwell-like tensors coincides with that of the four-derivative action – see (3.5). Moreover, the expression (3.28) shows clearly that the action actually admits a larger gauge symmetry: that of the four-derivative Maxwell-like action  $\mathcal{M}_4$  (2.30, 2.34) rather than the Fronsda one with traceless parameter.

### General case

The  $(4n+2)$ -derivative Weyl-like action can be obtained generalizing the six-derivative one (3.28) to

$$\mathcal{W}_{4n+2}[\varphi^{(s)}] = \langle\langle G_{2n}^{(s)} \mid \mathcal{A}_{2n} M_{2n+2}^{(s)} \rangle\rangle, \quad (3.29)$$

where the gauge field is subject to the  $(n+1)$ -th traceless constraint. The kinetic operator in the above action is self-adjoint: first, the Maxwell-like tensor can be written as  $M_{2n+2}^{(s)} = \mathcal{K}_2 G_{2n}^{(s)}$  with a two-derivative operator  $\mathcal{K}_2$ , then  $\mathcal{A}_{2n} \mathcal{K}_2$  can be recast into a manifestly self-adjoint form using the Bianchi-like identities (2.17). From the self-adjointness, one can see that the gauge symmetry of (3.29) is that of the  $(2n+2)$ -derivative Maxwell-like action  $\mathcal{M}_{2n+2}$ . Moreover, the Weyl variation of (3.29) reads simply

$$\delta_\alpha \mathcal{W}_{4n+2}[\varphi^{(s)}] = 2 \langle\langle \delta_\alpha G_{2n}^{(s)} \mid \mathcal{A}_{2n} M_{2n+2}^{(s)} \rangle\rangle, \quad (3.30)$$

so that the analysis goes along the same lines as in the  $4n$ -derivative case: requiring the absence of  $c_k^\sharp$  terms in the Weyl variation (3.16), the operator  $\mathcal{A}_{2n}$  is completely determined by (3.18). The remained  $c_k^\flat$  terms are proportional to double gradients, so that integrating by parts one gradient one gets a divergence of Maxwell-like tensor,  $\partial_u \cdot \partial_x M_{2n+2}^{(s)}$ . The latter vanishes (differently from the  $4n$ -derivative case where one gets  $\partial_u \cdot \partial_x G_{2n}^{(s)}$ ) when it

is contracted with a tensor whose divergence of the  $n$ -th trace vanishes – see (2.30), so that the constraint on the Weyl parameter finally reads

$$(\partial_u \cdot \partial_x)^2 (\partial_u^2)^{n-1} \mathcal{F}_{2(n-1)} \cdots \mathcal{F}_2 \alpha^{(s-2)} = 0. \quad (3.31)$$

After using the gauge-for-gauge freedom to reach the traceless condition (3.22), one gets the multiple divergence constraint

$$(\partial_x \cdot \partial_u)^{2n} \alpha^{(s-2)}(x, u) = 0. \quad (3.32)$$

The Skvortsov-Vasiliev action [30] is a member of this hierarchy with  $n = 0$ .

### 3.3 Emergence of Francia's mass term

In linearized Weyl gravity (or more generally in the four-derivative Weyl-like actions), the trace modifier appearing between two Einstein tensors turns out to coincide with the inverse of the Fierz-Pauli mass term:

$$(\mathcal{A}_2)^{-1} = \mathcal{I}_{\text{FP}} = 1 - \frac{1}{2} u^2 \partial_u^2. \quad (3.33)$$

Therefore, one may expect that the trace modifiers  $\mathcal{A}_{2n}$ 's appearing in the Weyl-like actions be also inverses of some mass operators. In order to check this idea, one may explicitly compute the inverse of  $\mathcal{A}_{2n}$ :

$$(\mathcal{A}_{2n})^{-1} = \mathcal{I}_{2n} (\mathcal{C}_{2n})^{-1}. \quad (3.34)$$

However, there is a shortcut. Instead of directly computing  $(\mathcal{A}_{2n})^{-1}$ , we first conjecture that they all coincide with the higher-spin analogue of Fierz-Pauli mass term introduced by Francia in [33],

$$\mathcal{I}_{\text{F}} = 1 - \frac{1}{2} u^2 \partial_u^2 - \frac{1}{8} (u^2)^2 (\partial_u^2)^2 - \cdots - \frac{1}{2^k k! (2k-3)!!} (u^2)^k (\partial_u^2)^k - \cdots, \quad (3.35)$$

which is determined by the property:

$$\partial_x \cdot \partial_u \mathcal{I}_{\text{F}} = \tilde{\mathcal{I}}_{\text{F}} (\partial_x \cdot \partial_u - u \cdot \partial_x \partial_u^2). \quad (3.36)$$

Here  $\tilde{\mathcal{I}}_{\text{F}}$  is some operator of the same type as  $\mathcal{I}_{\text{F}}$ . Rewriting the relation to the mass operator in a different way

$$\mathcal{I}_{\text{F}} = \mathcal{I}_{2n} (\mathcal{C}_{2n})^{-1} \quad \Leftrightarrow \quad \mathcal{I}_{\text{F}} \mathcal{C}_{2n} = \mathcal{I}_{2n}, \quad (3.37)$$

the property (3.36) of  $\mathcal{I}_{\text{F}}$ , together with the property (2.21)<sup>5</sup> of  $\mathcal{I}_{2n}$ , induces a condition on the operator  $\mathcal{C}_{2n}$  of (3.10):

$$(\partial_x \cdot \partial_u - u \cdot \partial_x \partial_u^2) \mathcal{C}_{2n} = \tilde{\mathcal{C}}_{2n} (\partial_x \cdot \partial_u - \frac{1}{2n} u \cdot \partial_x \partial_u^2), \quad (3.38)$$

---

<sup>5</sup> From the conditions (2.21) and (3.36), one can see that the Francia mass operator actually belongs to the class of the trace modifiers  $\mathcal{I}_n$  as  $\mathcal{I}_{\text{F}} = \mathcal{I}_1$ .

where  $\tilde{\mathcal{C}}_{2n}$  is again some operator of the same type as  $\mathcal{C}_{2n}$ . This condition determines completely the operator  $\mathcal{C}_{2n}$  with  $c_0 = 1$ . To see how this works, let us first consider the relations

$$\begin{aligned}\partial_x \cdot \partial_u \mathcal{C}_{2n} &= \sum_{k=0}^{\infty} (u^2)^{k-1} (\partial_u^2)^{k-1} \left\{ 2k c_k u \cdot \partial_x \partial_u^2 + [c_{k-1} + 4k(k-1)c_k] \partial_x \cdot \partial_u \right\}, \\ u \cdot \partial_x \partial_u^2 \mathcal{C}_{2n} &= \sum_{k=0}^{\infty} (u^2)^{k-1} (\partial_u^2)^{k-1} [c_{k-1} + 2k(d + 2u \cdot \partial_u - 2k)c_k] \times \\ &\quad \times [u \cdot \partial_x \partial_u^2 - 2(k-1)\partial_x \cdot \partial_u],\end{aligned}\tag{3.39}$$

where the  $c_k$ 's are the coefficients appearing in the expansion of  $\mathcal{C}_{2n}$  (3.10). The left-hand side of the difference between two equations in (3.39) coincides with (3.38). Focusing on the right-hand side, the terms proportional to  $\partial_x \cdot \partial_u$  and  $u \cdot \partial_x \partial_u^2$  give respectively

$$\begin{aligned}\partial_x \cdot \partial_u &\Rightarrow \tilde{c}_{k-1} = (2k-1)c_{k-1} - 4k(k-1)(d + 2s - 2k - 3)c_k, \\ u \cdot \partial_x \partial_u^2 &\Rightarrow \frac{1}{2n} \tilde{c}_{k-1} = c_{k-1} + 2k(d + 2s - 2k - 3)c_k,\end{aligned}\tag{3.40}$$

where  $\tilde{c}_k$ 's are the coefficients of  $\tilde{\mathcal{C}}_{2n}$ . Solving for the  $\tilde{c}_k$ 's from the above equations, one ends up with a recurrence relation between  $c_{k-1}$  and  $c_k$ , which exactly coincides with (3.18). This proves the conjecture  $(\mathcal{A}_{2n})^{-1} = \mathcal{I}_F$ . Let us comment here that the tensors  $S_{2n}^{(s)} := \mathcal{I}_F^{-1} G_{2n}^{(s)}$  can be regarded as generalizations of Schouten tensor in the sense that they transform under the Weyl transformation as double gradient:  $\delta_\alpha S_{2n}^{(s)} = (u \cdot \partial_x)^2 (\dots)$ .

## 4 Discussion

In this paper, we have constructed higher-derivative actions for higher spins which are gauge and Weyl invariant with some constraints. For a given spin  $s$ , we have first considered Einstein-like and Maxwell-like actions involving from 2 to  $s$  derivatives. These actions proved essential for the construction of Weyl-like actions. The latter are associated with  $(n+1)$ -th traceless gauge fields, such that  $(\partial_u^2)^{n+1} \varphi^{(s)} = 0$ , and consist of two classes:

- The first is the  $4n$ -derivative one given by

$$\begin{aligned}\mathcal{W}_{4n}[\varphi^{(s)}] &= \langle\langle G_{2n}^{(s)} | \mathcal{I}_F^{-1} G_{2n}^{(s)} \rangle\rangle \\ &= \langle\langle \varphi^{(s)} | \mathcal{F}_2^\dagger \cdots \mathcal{F}_{2n}^\dagger \mathcal{I}_{2n} \mathcal{I}_F^{-1} \mathcal{I}_{2n} \mathcal{F}_{2n} \cdots \mathcal{F}_2 \varphi^{(s)} \rangle\rangle,\end{aligned}\tag{4.1}$$

where  $\mathcal{F}_n$ ,  $\mathcal{I}_{2n}$  and  $\mathcal{I}_F$  are given in (2.15), (2.22) and (3.35). This action is invariant under the gauge and Weyl transformations (2.2) with

$$(\partial_u^2)^n \varepsilon^{(s-1)} = 0, \quad \partial_u^2 \alpha^{(s-2)} = 0 = (\partial_u \cdot \partial_x)^{2n-1} \alpha^{(s-2)}.\tag{4.2}$$

- The second is the  $(4n+2)$ -derivative one given by

$$\begin{aligned}\mathcal{W}_{4n+2}[\varphi^{(s)}] &= \langle\langle G_{2n}^{(s)} | \mathcal{I}_F^{-1} M_{2n+2}^{(s)} \rangle\rangle \\ &= \langle\langle \varphi^{(s)} | \mathcal{F}_2^\dagger \cdots \mathcal{F}_{2n}^\dagger \mathcal{I}_{2n} \mathcal{I}_F^{-1} \mathcal{I}_{2n+2} \mathcal{F}_{2n+1} \cdots \mathcal{F}_1 \mathcal{L} \varphi^{(s)} \rangle\rangle,\end{aligned}\tag{4.3}$$



which is invariant under the gauge and Weyl transformations (2.2) with

$$(\partial_u^2)^{n+1} \varepsilon^{(s-1)} = 0 = \partial_x \cdot \partial_u (\partial_u^2)^n \varepsilon^{(s-1)}, \quad \partial_u^2 \alpha^{(s-2)} = 0 = (\partial_u \cdot \partial_x)^{2n} \alpha^{(s-2)}. \quad (4.4)$$

Notice that for  $s = 2n$  or  $2n + 1$ , all constraints are dropped, and the action  $\mathcal{W}_{2s}[\varphi^{(s)}]$  coincides with the higher-spin Weyl action.

The Weyl-like actions  $\mathcal{W}_{2n}$ , being higher-derivative, contain ghost modes in the spectrum and, as a result, lead to non-unitary representations of the Poincaré algebra. Although non-unitary, however, these may still exhibit interesting mathematical properties. Let us recall that the spin-two Weyl action propagates, around a flat background, two (relatively ghost) helicity two modes and a helicity one mode [35]. Interestingly, analyzed around an (A)dS background, the above spectrum groups into two packages: a massless spin two and a partially-massless spin two [15, 21]. Diagrammatically, this can be expressed as

$$\mathcal{W}_4[\varphi^{(2)}] \Rightarrow \begin{array}{c} + \\ 2 \end{array} \quad \begin{array}{c} - \\ \boxed{2} \\ 1 \end{array} \quad (4.5)$$

where each number indicates the corresponding helicity mode while the enclosure means that the helicities therein become an irreducible set in (A)dS as pertains to a partially-massless representation [36]. Moreover, the analysis of the spectrum for spin- $s$  Weyl action around a flat background shows that it propagates  $\ell$  copies of helicity- $\ell$  modes with  $\ell = s, s-1, \dots, 1$  [23]. From the analogy of the spin two case, it is natural to expect that, when deformed to an (A)dS background, all these spectra group into partially-massless spin- $s$  modes with alternating signs of their kinetic operators. As in the spin two case, this can be summarized via the following diagram

$$\mathcal{W}_{2s}[\varphi^{(s)}] \Rightarrow \begin{array}{c} + \\ s \end{array} \quad \begin{array}{c} - \\ \boxed{s} \\ s-1 \end{array} \quad \dots \quad \begin{array}{c} \pm \\ \boxed{s} \\ s-1 \\ \vdots \\ 2 \end{array} \quad \begin{array}{c} \mp \\ \boxed{s} \\ s-1 \\ \vdots \\ 2 \\ 1 \end{array} \quad (4.6)$$

where the  $r$ -th block corresponds to partially-massless spin  $s$  of the  $r$ -th point (or, equivalently, of depth  $r$ ). While each block is irreducible under the (A)dS isometry group, the entire spectrum provides an indecomposable representation of the conformal group [23]. This gives a hint for a novel class of non-unitary but interesting representations of the conformal group, covering all short representations of the isometry group.

One may expect that there exist even a  $(2r+2)$ -derivative action propagating partially-massless fields from the zero-th (massless) point to the  $r$ -th point.

$$\begin{array}{c} + \\ s \end{array} \quad \begin{array}{c} - \\ \boxed{s} \\ s-1 \end{array} \quad \dots \quad \begin{array}{c} \pm \\ \boxed{s} \\ s-1 \\ \vdots \\ s-r \end{array} \quad (4.7)$$

If this action exists, the (A)dS deformation of the Weyl-like action  $\mathcal{W}_{2r+2}$  can be a good candidate. In fact, it is the case for  $r = 1$ : the four-derivative Weyl-like action around (A)dS can be obtained as

$$\mathcal{W}_4^\Lambda[\varphi^{(s)}] = \langle\langle G_2^{\Lambda(s)} | \mathcal{I}_F^{-1} G_2^{\Lambda(s)} \rangle\rangle - \eta \Lambda \langle\langle \varphi^{(s)} | G_2^{\Lambda(s)} \rangle\rangle \quad \left[ \eta = \frac{2(d+2s-6)}{(d-1)(d-2)} \right], \quad (4.8)$$

adding a Fronsdal action to the four-derivative part. Here,  $G_2^{\Lambda(s)} = \mathcal{I}_2 \mathcal{F}_2^\Lambda \varphi^{(s)}$  is the spin- $s$  cosmological Einstein tensor, and the field  $\varphi^{(s)}(x, u)$  is contracted with flat auxiliary variables  $u^\alpha$ 's and the AdS vielbein  $\bar{e}_\alpha^\mu$  as

$$\varphi(x, u) = \frac{1}{s!} u^{\alpha_1} \cdots u^{\alpha_s} \bar{e}_{\alpha_1}^{\mu_1}(x) \cdots \bar{e}_{\alpha_s}^{\mu_s}(x) \varphi_{\mu_1 \cdots \mu_s}(x). \quad (4.9)$$

The Fronsdal operator  $\mathcal{F}_2^\Lambda$  in (A)dS is given, in terms of the covariant derivative

$$D_\alpha = \bar{e}_\alpha^\mu \nabla_\mu + \frac{1}{2} \bar{\omega}_{\alpha\beta}^\gamma u^\beta \partial_{u^\gamma}, \quad (4.10)$$

by

$$\mathcal{F}_2^\Lambda = D^2 - u \cdot D \partial_u \cdot D + \frac{1}{2} (u \cdot D)^2 \partial_u^2 + \frac{2\Lambda}{(d-1)(d-2)} [u^2 \partial_u^2 + s^2 + (d-6)s - 2(d-3)]. \quad (4.11)$$

Similarly to the spin two case, this action can be recast in the form:

$$\mathcal{W}_4^\Lambda = \eta \Lambda [-\langle\langle \varphi^{(s)} | G_2^\Lambda(\varphi^{(s)}) \rangle\rangle + \langle\langle \chi^{(s)} | G_2^\Lambda(\chi^{(s)}) \rangle\rangle - \eta \Lambda \mathcal{I}_F \chi^{(s)}], \quad (4.12)$$

where the first term corresponds to the spin- $s$  Fronsdal action while the second describes the partially-massless spin  $s$  of the first point: the latter admits a gauge description via doubly-traceless tensors  $\chi^{(s)}$  and  $\chi^{(s-1)}$ , with corresponding traceless gauge parameters  $\alpha^{(s-1)}$  and  $\alpha^{(s-2)}$ . After gauge fixing the Stueckelberg field  $\chi^{(s-1)}$  using  $\alpha^{(s-1)}$  and  $\partial_u \cdot D \alpha^{(s-2)}$ , one ends up with the system in (4.12) with the residual gauge symmetry:

$$\delta_\alpha \chi^{(s)} = [(u \cdot D)^2 + \Lambda u^2] \alpha^{(s-2)}, \quad \partial_u \cdot D \alpha^{(s-2)} = 0 = \partial_u^2 \alpha^{(s-2)}. \quad (4.13)$$

For more details, see e.g. [36–38]. Let us notice that the transversality constraint on the Weyl parameter  $\alpha^{(s-2)}$  of  $\mathcal{W}_4^\Lambda$  arises by a partial gauge fixing procedure, as the transversality constraints on  $\varepsilon^{(s-1)}$  of  $\mathcal{M}_{2n}$  (which can be obtained by partially gauge fixing  $\mathcal{G}_{2n}$ ). Hence, there may exist an action without any differential constraint on the Weyl parameter but involving auxiliary fields, such that gauge fixing all the auxiliary fields leads to  $\mathcal{W}_{2n}$ . Such an action may have an ordinary derivative formulation, involving an off-shell field for each propagating degree of freedom, analogous to those considered in [23, 38].

The problem of non-unitarity in these theories can be in principle handled as in the gravity case:

- In AdS background, one can select only the massless spin  $s$ , as in [15], making use of suitable boundary conditions.
- In three dimensions, ghosts do not propagate assuming the spectrum (4.7): Weyl action  $\mathcal{W}_{2s}$  propagates a single scalar mode, while the other members do not have any propagating content. Hence, it would be interesting to consider these actions in the context of AdS<sub>3</sub>/CFT<sub>2</sub> correspondence.

To conclude, let us discuss briefly the generalization of our actions to interacting ones. Despite many efforts, no deformation of AdS Fronsdal action to fully interacting one is available, while Vasiliev's equations [39, 40] describe propagation of an infinite tower of massless interacting higher spins. On the other hand, the linear conformal higher-spin action ( $d = 4$  Weyl action) can be deformed into a fully non-linear action<sup>6</sup> without auxiliary fields [43, 44] but with the entire tower of higher-spin fields. From the conformal gravity, one can obtain the Einstein action at the level of interacting theory [14, 15].<sup>7</sup> Hopefully, a similar mechanism can also work for the higher-spin case, providing new insights for a more conventional action principle<sup>8</sup> leading to Vasiliev's equations.

## Acknowledgments

We would like to thank Luca Lopez, Massimo Taronna, Andrew Waldron and especially Dario Francia and Augusto Sagnotti for fruitful discussions and useful comments. This work was supported in part by Scuola Normale Superiore, by INFN (I.S. TV12) and by the MIUR-PRIN contract 2009-KHZKRX. The work of KM is also supported by the ERC Advanced Investigator Grants no. 226455 ‘‘Supersymmetry, Quantum Gravity and Gauge Fields’’ (SUPERFIELDS).

## A Spin three Weyl action

In this section, we construct spin three Weyl action in a form which is suitable for the generalization to six-derivative Weyl-like action for any spin.

The variation of the spin three Fronsdal tensor with respect to gauge transformation  $\delta\varphi_{\mu\nu\rho} = 3\partial_{(\mu}\varepsilon_{\nu\rho)}$  gives

$$\delta F_{\mu\nu\rho} = 3\partial_\mu\partial_\nu\partial_\rho\varepsilon_\sigma{}^\sigma, \quad (\text{A.1})$$

and therefore the following tensor (antisymmetric with respect to first two indices and symmetric with respect to second two):

$$C_{\mu\nu,\rho\sigma} = \partial_\mu F_{\nu\rho\sigma} - \partial_\nu F_{\mu\rho\sigma} \quad (\text{A.2})$$

is gauge invariant. The spin three Weyl Lagrangian can be conveniently expressed in terms of  $C_{\mu\nu,\rho\sigma}$  as

$$\mathcal{L} = -\frac{1}{2}C_{\mu\nu,\rho\sigma}C^{\mu\nu,\rho\sigma} + \frac{d+4}{8(d+1)}C'_{\mu\nu}C'^{\mu\nu} \quad [C'_{\mu\nu} = C_{\mu\nu,\rho}{}^\rho]. \quad (\text{A.3})$$

Up to integration by parts, the above Lagrangian can be recast into this form

$$\mathcal{L} = F_{\mu\nu\rho}\square F^{\mu\nu\rho} - \frac{d+4}{2(d+1)}F'_\mu\square F'^\mu - \frac{d-2}{2(d+1)}F^{\mu\nu\rho}\partial_{(\mu}\partial^\sigma F_{\nu\rho)\sigma} \quad [F'_\mu = F_{\mu\nu}{}^\nu], \quad (\text{A.4})$$

which has been used in (3.24) for generalization to any spin.

---

<sup>6</sup> The conformal higher-spin action, analogously to the gravity case, might make use of non-linear Weyl tensor – (appropriately defined) traceless part of non-linear higher-spin curvature. See [41] for an attempt on the non-linear deformation of the curvature. See also [42] for the cubic interaction in the frame-like approach.

<sup>7</sup> On the other hand, it has been shown that extracting the other component of conformal gravity – partially-massless spin two – faces a consistency problem associated with the interaction structure [21].

<sup>8</sup> See also the recent proposal [45] and references therein.

## B Spectrum of the four-derivative gauge invariant action

In Section 4, we analyzed the spectrum of the four-derivative Weyl-like action. To complete the study of the spectrum of four-derivative actions considered in this paper, we derive here the spectrum of the four-derivative Einstein-like action. Since the Maxwell-like actions can be obtained by partially gauge fixing the Einstein-like ones, the spectrum of the two should coincide (as in the case of SV and Fronsdal actions).

The four-derivative Einstein-like action  $\mathcal{G}_4$  gives the equation of motion:

$$0 = F_4^{(s)} := \mathcal{F}_4 \mathcal{F}_2 \varphi^{(s)} = [(\partial_x^2)^2 - u \cdot \partial_x \mathcal{D}] \varphi^{(s)}, \quad (\text{B.1})$$

where

$$\mathcal{D} = \partial_x^2 \partial_x \cdot \partial_u - \frac{1}{6} u \cdot \partial_x [\partial_x^2 \partial_u^2 + 2(\partial_x \cdot \partial_u)^2] + \frac{1}{6} (u \cdot \partial_x)^2 \partial_x \cdot \partial_u \partial_u^2 - \frac{1}{24} (u \cdot \partial_x)^3 (\partial_u^2)^2. \quad (\text{B.2})$$

The three-derivative operator  $\mathcal{D}$  satisfies the following properties:

$$(\partial_u^2)^2 \mathcal{D} \varphi^{(s)} = 0, \quad \delta_\varepsilon (\mathcal{D} \varphi^{(s)}) = (\partial_x^2)^2 \varepsilon^{(s-1)}, \quad (\text{B.3})$$

analogously to the de Donder operator in Fronsdal's theory. In particular, the second property implies that the gauge condition

$$\mathcal{D} \varphi^{(s)} = 0, \quad (\text{B.4})$$

is an allowed one. This gauge condition gives for the gauge field and the parameter (associated with the residual gauge symmetry) the equations

$$(\partial_x^2)^2 \varphi^{(s)} = 0, \quad (\partial_x^2)^2 \varepsilon^{(s-1)} = 0, \quad (\text{B.5})$$

which are subject to the trace constraints

$$(\partial_u^2)^3 \varphi^{(s)} = 0, \quad (\partial_u^2)^2 \varepsilon^{(s-1)} = 0. \quad (\text{B.6})$$

Now we need first to solve these equations and fix the residual gauge symmetries. We can then plug back the solutions into (B.4) to obtain the on-shell constraints. Solving the latter will yield the spectrum of the theory.

The solution of the gauge fixed equation (B.5) is

$$\varphi^{(s)}(x, u) = \int d^d p \, \delta(p^2) [\tilde{\varphi}_1^{(s)}(p, u) + n \cdot x \tilde{\varphi}_2^{(s)}(p, u)] e^{ip \cdot x}, \quad (\text{B.7})$$

where  $n^\mu$  is a time-like vector which we choose  $n^\mu = \delta_0^\mu$ . The Fourier mode  $\tilde{\varphi}_1^{(s)}$  corresponds to the regular solutions, while  $\tilde{\varphi}_2^{(s)}$  to the ghost ones. After a suitable Lorentz transformation, the momentum can be tuned to  $(p_+, p_-, p_i) = (k_+, 0, 0)$  in light-cone coordinates:

$$u^\pm = \frac{1}{\sqrt{2}} (u^0 \pm u^{d-1}), \quad i = 1, \dots, d-2. \quad (\text{B.8})$$

After solving the same equation for the gauge parameter, we further gauge fix on-shell to

$$\tilde{\varphi}_{1,2}^{(s)}(k; u^+, u^-, u^i) = f_{1,2}^{(s)}(k; u^-, u^i) + u^+ (u_i^2)^2 \tilde{\varphi}_{1,2}^{(s-5)}(k; u^+, u^-, u^i). \quad (\text{B.9})$$

Note that we cannot gauge fix to zero all the components proportional to  $u^+$  due to the trace constraints (B.6). Plugging the solution (B.7) into the gauge condition (B.4), yields the two equations:

$$k_+ u^+ \left[ \partial_{u-}^2 - \frac{1}{2} u^+ \partial_{u-} \partial_u^2 + \frac{1}{8} (u^+)^2 (\partial_u^2)^2 \right] \tilde{\varphi}_1^{(s)} + \left[ \partial_{u-} - \frac{1}{6} u^+ \partial_u^2 + 3 u^+ (\partial_u^2 - \frac{1}{2} u^+ \partial_{u-} \partial_u^2) + \frac{3}{8} (u^+)^3 (\partial_u^2)^2 \right] \tilde{\varphi}_2^{(s)} = 0, \quad (\text{B.10})$$

$$\left[ \partial_{u-}^2 - \frac{1}{2} u^+ \partial_{u-} \partial_u^2 + \frac{1}{8} (u^+)^3 (\partial_u^2)^2 \right] \tilde{\varphi}_2^{(s)} = 0. \quad (\text{B.11})$$

Using (B.9) and  $\partial_u^2 = \partial_{u+} \partial_{u-} + \partial_{u^i}^2$ , then give

$$\begin{aligned} \tilde{\varphi}_{1,2}^{(s-5)} &= 0, & (\partial_{u^i}^2)^2 f_{1,2}^{(s)} &= 0, & \partial_{u-} f_2^{(s)} &= 0, \\ \partial_{u-} \partial_{u^i}^2 f_1^{(s)} &= 0, & \partial_{u-}^2 f_1^{(s)} &= \frac{1}{12 k_+} \partial_{u^i}^2 f_2^{(s)}, & \partial_{u-}^3 f_1^{(s)} &= 0. \end{aligned} \quad (\text{B.12})$$

Finally, one can identify the  $so(d-2)$  polarization tensors of propagating modes:

$$\begin{aligned} f_1^{(s)}(k; u^-, u^i) &= \theta_1^{(s)}(k; u^i) + (u^i)^2 \theta_1^{(s-2)}(k; u^i) + u^- \theta_1^{(s-1)}(k; u^i) + (u^-)^2 \frac{d+2s-6}{12 k_+} \theta_2^{(s-2)}(k; u^i), \\ f_2^{(s)}(k; u^-, u^i) &= \theta_2^{(s)}(k; u^i) + (u^i)^2 \theta_2^{(s-2)}(k; u^i), \end{aligned} \quad (\text{B.13})$$

which correspond to spin  $s, s-1, s-2$  massless regular modes and spin  $s, s-2$  massless ghosts.

## References

- [1] B. de Wit and D. Z. Freedman, *Systematics of Higher Spin Gauge Fields*, *Phys. Rev.* **D21** (1980) 358.
- [2] C. Fronsdal, *Massless Fields with Integer Spin*, *Phys. Rev.* **D18** (1978) 3624.
- [3] D. Francia and A. Sagnotti, *Minimal local Lagrangians for higher-spin geometry*, *Phys. Lett.* **B624** (2005) 93–104, [[hep-th/0507144](#)].
- [4] I. Buchbinder, A. Galajinsky, and V. Krykhtin, *Quartet unconstrained formulation for massless higher spin fields*, *Nucl. Phys.* **B779** (2007) 155–177, [[hep-th/0702161](#)].
- [5] D. Francia and A. Sagnotti, *Free geometric equations for higher spins*, *Phys. Lett.* **B543** (2002) 303–310, [[hep-th/0207002](#)].
- [6] X. Bekaert, S. Cnockaert, C. Iazeolla, and M. Vasiliev, *Nonlinear higher spin theories in various dimensions*, [[hep-th/0503128](#)].
- [7] X. Bekaert, N. Boulanger, and P. Sundell, *How higher-spin gravity surpasses the spin two barrier: no-go theorems versus yes-go examples*, [arXiv:1007.0435](#).
- [8] A. Sagnotti, *Notes on Strings and Higher Spins*, [arXiv:1112.4285](#).
- [9] K. Stelle, *Renormalization of Higher Derivative Quantum Gravity*, *Phys.Rev.* **D16** (1977) 953–969.
- [10] E. A. Bergshoeff, O. Hohm, and P. K. Townsend, *Massive Gravity in Three Dimensions*, *Phys.Rev.Lett.* **102** (2009) 201301, [[arXiv:0901.1766](#)].
- [11] E. A. Bergshoeff, O. Hohm, and P. K. Townsend, *On Higher Derivatives in 3D Gravity and Higher Spin Gauge Theories*, *Annals Phys.* **325** (2010) 1118–1134, [[arXiv:0911.3061](#)].

- [12] E. A. Bergshoeff, M. Kovacevic, J. Rosseel, P. K. Townsend, and Y. Yin, *A spin-4 analog of 3D massive gravity*, *Class.Quant.Grav.* **28** (2011) 245007, [[arXiv:1109.0382](#)].
- [13] E. A. Bergshoeff, J. Fernandez-Melgarejo, J. Rosseel, and P. K. Townsend, *On 'New Massive' 4D Gravity*, *JHEP* **1204** (2012) 070, [[arXiv:1202.1501](#)].
- [14] R. Metsaev, *Ordinary-derivative formulation of conformal low spin fields*, *JHEP* **1201** (2012) 064, [[arXiv:0707.4437](#)].
- [15] J. Maldacena, *Einstein Gravity from Conformal Gravity*, [arXiv:1105.5632](#).
- [16] H. Lu, Y. Pang, and C. Pope, *Conformal Gravity and Extensions of Critical Gravity*, *Phys.Rev.* **D84** (2011) 064001, [[arXiv:1106.4657](#)].
- [17] S.-J. Hyun, W.-J. Jang, J.-H. Jeong, and S.-H. Yi, *Noncritical Einstein-Weyl Gravity and the AdS/CFT Correspondence*, *JHEP* **1201** (2012) 054, [[arXiv:1111.1175](#)].
- [18] K. Stelle, *Classical Gravity with Higher Derivatives*, *Gen.Rel.Grav.* **9** (1978) 353–371.
- [19] S. Lee and P. van Nieuwenhuizen, *Counting of states in higher derivative field theories*, *Phys.Rev.* **D26** (1982) 934.
- [20] I. Buchbinder and S. Lyakhovich, *Canonical quantization and local measure  $R^2$  gravity*, *Class.Quant.Grav.* **4** (1987) 1487–1501.
- [21] S. Deser, E. Joung, and A. Waldron, *Partial Masslessness and Conformal Gravity*, [arXiv:1208.1307](#).
- [22] E. S. Fradkin and A. A. Tseytlin, *Conformal Supergravity*, *Phys. Rept.* **119** (1985) 233–362.
- [23] R. Metsaev, *Ordinary-derivative formulation of conformal totally symmetric arbitrary spin bosonic fields*, *JHEP* **1206** (2012) 062, [[arXiv:0709.4392](#)].
- [24] R. Marnelius, *Lagrangian conformal higher spin theory*, [arXiv:0805.4686](#).
- [25] O. Shaynkman, I. Y. Tipunin, and M. Vasiliev, *Unfolded form of conformal equations in  $M$  dimensions and  $o(M + 2)$  modules*, *Rev.Math.Phys.* **18** (2006) 823–886, [[hep-th/0401086](#)].
- [26] R. Metsaev, *Shadows, currents and AdS*, *Phys.Rev.* **D78** (2008) 106010, [[arXiv:0805.3472](#)].
- [27] R. Metsaev, *Gauge invariant two-point vertices of shadow fields, AdS/CFT, and conformal fields*, *Phys. Rev.* **D81** (2010) 106002, [[arXiv:0907.4678](#)].
- [28] M. Vasiliev, *Bosonic conformal higher-spin fields of any symmetry*, *Nucl.Phys.* **B829** (2010) 176–224, [[arXiv:0909.5226](#)].
- [29] X. Bekaert and M. Grigoriev, *Notes on the ambient approach to boundary values of AdS gauge fields*, [arXiv:1207.3439](#).
- [30] E. Skvortsov and M. Vasiliev, *Transverse Invariant Higher Spin Fields*, *Phys.Lett.* **B664** (2008) 301–306, [[hep-th/0701278](#)].
- [31] A. Campoleoni and D. Francia, *Maxwell-like Lagrangians for higher spins*, [arXiv:1206.5877](#).
- [32] D. Francia, *Generalised connections and higher-spin equations*, *Class. Quant. Grav.* **29** (2012) 245003, [[arXiv:1209.4885](#)].
- [33] D. Francia, *Geometric Lagrangians for massive higher-spin fields*, *Nucl.Phys.* **B796** (2008) 77–122, [[arXiv:0710.5378](#)].
- [34] D. Francia, *String theory triplets and higher-spin curvatures*, *Phys.Lett.* **B690** (2010) 90–95, [[arXiv:1001.5003](#)].

- [35] R. Riegert, *The particle content of linearized conformal gravity*, *Phys.Lett.* **A105** (1984) 110–112.
- [36] S. Deser and A. Waldron, *Partial masslessness of higher spins in (A)dS*, *Nucl.Phys.* **B607** (2001) 577–604, [[hep-th/0103198](#)].
- [37] Y. Zinoviev, *On massive high spin particles in AdS*, [hep-th/0108192](#).
- [38] R. Metsaev, *CFT adapted gauge invariant formulation of massive arbitrary spin fields in AdS*, *Phys.Lett.* **B682** (2010) 455–461, [[arXiv:0907.2207](#)].
- [39] M. A. Vasiliev, *Consistent equation for interacting gauge fields of all spins in (3+1)-dimensions*, *Phys. Lett.* **B243** (1990) 378–382.
- [40] M. Vasiliev, *Higher spin gauge theories in any dimension*, *Comptes Rendus Physique* **5** (2004) 1101–1109, [[hep-th/0409260](#)].
- [41] R. Manvelyan, K. Mkrtchyan, W. Ruhl, and M. Tovmasyan, *On Nonlinear Higher Spin Curvature*, *Phys.Lett.* **B699** (2011) 187–191, [[arXiv:1102.0306](#)].
- [42] E. Fradkin and V. Linetsky, *Cubic interaction in conformal theory of integer higher spin fields in four-dimensional space-time*, *Phys. Lett.* **B231** (1989) 97.
- [43] A. Y. Segal, *Conformal higher spin theory*, *Nucl. Phys.* **B664** (2003) 59–130, [[hep-th/0207212](#)].
- [44] X. Bekaert, E. Joung, and J. Mourad, *Effective action in a higher-spin background*, *JHEP* **1102** (2011) 048, [[arXiv:1012.2103](#)].
- [45] N. Boulanger and P. Sundell, *An action principle for Vasiliev’s four-dimensional higher-spin gravity*, [arXiv:1102.2219](#).